TWMS J. Pure Appl. Math., V.8, N.2, 2017, pp.170-185

CONTROLLABILITY OF PROCESS DESCRIBED BY LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this article we offer new methods for constructing the program and position control for processes described by linear ordinary differential equations with phase and integral restrictions and boundary conditions. We describe an algorithm for solving the optimal control problem. In this work we show existence for solution of a controllability problem. If the problem has a solution we describe methods for constructing the optimal control .

Keywords: dynamical system, integral systems, program control, position control, optimal high-speed performance.

AMS Subject Classification: 34C05, 34C07, 34C25.

1. Introduction

Controllability theory takes its beginning in work of R.E. Kalman [7]. He has constructed a control with minimal norm and obtained ranking criteria of controllability for linear systems with fixed parameters. Solution of the controllability problem based on the l-problem of moments was offered by N.N. Krasovskiy [9]. Certain issues of controllability: the smallest dimension of the control vector, controllability of nonlinear systems with a small parameter, controllability of linear systems with aftereffect have been studied in works [5, 12]. An introduction to modern linear control theory is given in [6].

In recent years there appered many scientific articles dedicated to the problems of controllability and optimal high-speed performance of dynamical systems. Synthesis of position control for linear dynamical systems with using of the functions of Lyapunov is proposed in [3]. A geometric approach for solving the controllability problem of nonautonomous linear systems is studied in [11]. Approximate solution for optimal control of linear systems with a quadratic performance index using the differential transformation method is given in [10].

The problem of controllability is closely connected to the solution of stabilization problems of dynamical systems. In one [4] considered the problem of stabilization for zero equilibrium state of bilinear and affine systems in canonical form. Minimum stabilizers for linear dynamical systems are studied in [8]. Problem of design of static output feedback controllers for stationary linear systems with continuous and discrete time are reviewed in [2]. In the mentioned works [2-11] one studied only special cases and not the general case. At the moment there are some urgent problems of optimal control that are not solved yet:

1) it is necessary to find necessary and sufficient conditions for the solvability the general problem of optimal control and high-speed performance

2) build constructive method for solving of the general problem of controllability and high-speed performance for ordinary differential equations.

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Manuscript received September 2016.

2. Statement of the problem

Consider the processes described by linear ordinary differential equations

$$\dot{x} = A(t)x + B(t)u + \mu(t),\ t \in I = [t_0, t_1], \tag{1}$$

with boundary conditions

$$(x(t_0) = x_0,\ x(t_1) = x_1) \in S_0 \times S_1 = S \subset R^{2n}, \tag{2}$$

with phase restrictions

$$x(t) \in G(t);\ G(t) = \{x \in R^n /\ \omega(t) \le L(t)x + l(t) \le \varphi(t), t \in I\}, \tag{3}$$

and integral restrictions

$$g_j(x,u) \le c_j,\ j = \overline{1,m_1};\ g_j(x,u) = c_j,\ j = \overline{m_1+1,m_2}, \tag{4}$$

$$g_j(x,u) = \int_{t_0}^{t_1} [< a_j(t), x > + < b_j(t), u >]dt,\ j = \overline{1,m_2}, \tag{5}$$

with the constraint on the control value

$$u(t) \in U(t) = \{u(\cdot) \in L_2(I,R^m)/\ u(t) \in V(t) \subset R^m \text{ a.e. } t \in I\}. \tag{6}$$

where $A(t),\ B(t)$ - matrices of $n \times n,\ n \times m$ orders respectively with piecewise continuous elements, $S_0,\ S_1$ - given bounded convex closed sets, $L(t),\ t \in I$ - given matrix of $s \times n$ order with piecewise continuous elements, $l(t),\ t \in I$ - a known vector-function $s \times 1$ with piecewise continuous elements, $\omega(t),\ \varphi(t),\ t \in I$ - given continuous vector-functions $s \times 1$, $a_j(t),\ b_j(t)$, $j = \overline{1,m_2}$ given piecewise continuous vector-functions of $n \times 1,\ m \times 1$ orders respectively, c_j, $j = \overline{1,m_2}$ - known constants, $V(t), t \in I$ - given convex closed set in R^m, $U = U(t), t \in I$ - given closed convex set of $L_2(I,R^m)$, $\mu(t),\ t \in I$ - given vector-function with piecewise continuous elements. Usually considers case when the dimension $m < n$.

Let us use next definitions for problem (1)-(6):

Definition 2.1. *If the equation $u(t) \in U,\ t \in I$, transfers the trajectory of the system (1) from the point $x_0 \in R^n$ to the point $x_1 \in R^n$ when $t_0,\ t_1 > t_0$ are fixed and satisfy conditions (2)-(6), then system (1) under conditions (2)-(6) is called controllable, and the control $u(t),\ t \in I$ is called the program control. If $u(t) = u(x(t),t) \in U$, then $u(x,t)$ is called position control.*

Definition 2.2. *Let t_0 be fixed, and value t_1 be not fixed. The pair $(u(t), x(t,u)),\ t \in I$ corresponding to the smallest value of t_1, which satisfies (1)-(6) is called a solution of the problem of optimal high-speed performance.*

Following problems are considered:

Proposition 2.1. *It is necessary to find program control $u(t) \in U(t), t \in I$, which transfers trajectory of the system (1) from initial point $x_0 = x(t_0) \in S_0 \subset R^n$ in moment t_0 to the point $x_1 = x(t_1) \in S_1 \subset R^n$, $t_1 > t_0$ and besides the solution of the system (1) functions $x(t) = x(t; t_0, x_0, u)$, $x_0 \in S_0$, $x_1 = x(t_1) \in S_1$, are from the set $G(t) \subset R^n$. Along the solutions of the system (1) must be performed integral constraints (4), (5).*

Proposition 2.2. *It is necessary to find positional control $u(x,t) \in U$ for the system (1) under the conditions (2)-(6).*

Proposition 2.3. *Let t_0 be a given time moment, value $t_1 > t_0$ is not fixed. It is necessary to find program control $u(t) \in U$ which transfer trajectory of the system (1) from initial point $x_0 = x(t_0) \in S_0$ to the given point $x_1 \in S_1$ in the shortest time $t_{1_*} - t_0$ under the conditions (2)-(6), where t_{1_*} - is the minimal value of t_1.*

Solving of the problems 2.1-2.3 is being actual both for mathematical control theory and many applied problems: management of nuclear and chemical reactors, control of spacecraft motion, management of multi-sector economic models and other.

3. Transformation

Consider the integral constraints (4), (5). Let us denote vector-function $\eta(t) = (\eta_1(t), \dots, \eta_{m_2}(t))$, $t \in I$ as follows

$$\eta_j(t) = \int_{t_0}^{t} [< a_j(\tau), x(\tau) > + < b_j(\tau), u(\tau) >] d\tau, \ j = \overline{1, m_2}, \ t \in I. \tag{7}$$

From (7) it follows that

$$\dot{\eta} = A_0(t)x + B_0(t)u(t), \ t \in I, \tag{8}$$

$$\begin{gathered} \eta(t_1) = \bar{c}, \ \bar{c} \in \Omega_1 = \{\bar{c} \in R^{m_2} / \ \bar{c}_j = c_j - d_j, \ j = \overline{1, m_1}, \bar{c}_j = c_j, \\ j = \overline{m_1 + 1, m_2}, \ d_j \geq 0, \ j = \overline{1, m_1}\}, \ \eta(t_0) = 0. \end{gathered} \tag{9}$$

Let's introduce the following vectors and matrices

$$\xi = \begin{pmatrix} x \\ \eta \end{pmatrix}, \ A_1(t) = \begin{pmatrix} A(t) & O_{nm_2} \\ A_0(t) & O_{m_2 m_2} \end{pmatrix}, \ B_1(t) = \begin{pmatrix} B(t) \\ B_0(t) \end{pmatrix}, \ \mu_1(t) = \begin{pmatrix} \mu(t) \\ O_{m_2 1} \end{pmatrix}$$

where $O_{k_1 q}$ - rectangular matrix of $k \times q$ order with zero elements

$$A_0(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_{m_2}(t) \end{pmatrix}, \ B_0(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_{m_2}(t) \end{pmatrix}, \ t \in I$$

- matrices of $m_2 \times n$, $m_2 \times m$ orders respectively

Now the relation (1)-(6) can be written in the following form

$$\dot{\xi} = A_1(t)\xi + B_1(t)u + \bar{\mu}_1(t), \ t \in I, \tag{10}$$

$$\xi(t_0) = \xi_0 = \begin{pmatrix} x(t_0) \\ \eta(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ O_{m_2 1} \end{pmatrix}, \ \xi(t_1) = \xi_1 = \begin{pmatrix} x(t_1) \\ \eta(t_1) \end{pmatrix} = \begin{pmatrix} x_1 \\ \bar{c} \end{pmatrix}, \tag{11}$$

$$(x_0, x_1) \in S_0 \times S_1, \ \xi(t_0) = \xi_0 \in S_0 \times O_{m_2 1}, \ \xi(t_1) = \xi_1 \in S_1 \times \Omega_1, \tag{12}$$

$$P\xi(t) = x(t) \in G(t); \ P = (I_n, O_{nm_2}), \ u(t) \in U(t), \ t \in I, \tag{13}$$

where I_n - unitary matrix of $n \times n$ order, $A_1(t)$, $B_1(t)$ - matrices of $(n+m_2) \times (n+m_2)$,$(n+m_2) \times m$ orders respectively, $\bar{\mu}(t)$ - is a known function $(n + m_2) \times 1$, function $\eta(t)$,$t \in I$ - is the solution of the equation (8), the set Ω_1 is described by (9).

Let us note, that relations (1)-(6) are equivalent to (10)-(13). Then problems 2.1-2.3 are equivalent to the following ones:

Proposition 3.1. *It is necessary to find program control $u(t) \in U(t)$, $t \in I$, which transfers trajectory of the system (10) from the initial point $\xi_0 = \xi(t_0) \in S_0 \times O_{m_2 1}$ in time t_0 to the point $\xi_1 = \xi(t_1) \in S_1 \times \Omega_1$,$t_1 > t_0$ besides the solution of the system (10) function $\xi(t) = \xi(t; t_0, \xi_0, u)$, $t \in I$ is such, that $P\xi(t) \in G(t)$, $t \in I$.*

Proposition 3.2. *It is necessary to find position control $u(P\xi, t) = u(P\xi(t), t) \in U$ for the system (10) under the conditions (11)-(13).*

Proposition 3.3. *It is necessary to find program control $u(t) \in U$ which transfers trajectory of the system (10) from the initial point $\xi_0 \in S_0 \times O_{m_2 1}$ to the given point $\xi_1 \in S_1 \times \Omega_1$ in minimal time $t_{1_*} - t_0$ with the conditions $P\xi(t) \in G(t)$, $t \in [t_0, t_{1_*}]$, where t_{1_*} - is the minimal value of t_1.*

4. Integral equation

To solve the problems of controllability and high-speed performance we use following theorems of properties of solutions of the integral Fredholm equation of the first kind from work [1]. Consider the integral equations of the following type

$$\int_{t_0}^{t_1} K(t_0, t)u(t)dt = a, \ t \in I = [t_0, t_1], \tag{14}$$

where $K(t_0, t) = \|K_{ij}(t_0, t)\|$, $i = \overline{1, n}$, $j = \overline{1, m}$ - is a known matrix of $n \times m$ order with piecewise continuous elements by t with fixed t_0,t_1, $u(\cdot) \in L_2(I, R^m)$ - unknown function, operator $K : L_2(I, R^m) \to R^n$, $a \in R^n$ is a given vector.

Theorem 4.1. *Integral equation (14) for any fixed $a \in R$ has a solution if and only if the matrix*

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t)K^*(t_0, t)dt \tag{15}$$

of $n \times n$ order is positively defined, where (*) - transposition sign, $t_1 > t_0$.

Theorem 4.2. *Let matrix $C(t_0, t_1)$ be positively defined. Then general solution of integral equation (14) has a form*

$$u(t) = K^*(t_0, t)C^{-1}(t_0, t_1)a + v(t) - K^*(t_0, t)C^{-1}(t_0, t_1) \int_{t_0}^{t_1} K(t_0, t)v(t)dt, \ t \in I, \tag{16}$$

where $v(\cdot) \in L_2(I, R^m)$ - arbitrary function, $a \in R^n$ - arbitrary vector.

Solution of the equation (14) has the following properties:

1) function $u(t)$, $t \in I$ from (16) can be represented as a sum $u(t) = u_1(t) + u_2(t)$, $t \in I$, where $u_1(t) = K^*(t_0, t)C^{-1}(t_0, t_1)a$, $t \in I$ - particular solution of the integral equation (14), $u_2(t) = v(t) - K^*(t_0, t) \cdot C^{-1}(t_0, t_1) \int_{t_0}^{t_1} K(t_0, t)v(t)dt$, $t \in I$, $\forall v(\cdot) \in L_2(I, R^m)$ - general solution of the homogeneous integral equation $\int_{t_0}^{t_1} K(t_0, t)u_2(t)dt = 0$;

2) functions $u_1(\cdot) \in L_2(I, R^m)$, $u_2(\cdot) \in L_2(I, R^m)$ orthogonal, i.e. $u_1 \perp u_2$, $< u_1, u_2 >_{L_2} = 0$;

3) function $u_1(t) = K^*(t_0, t)C^{-1}(t_0, t_1)a$, $t \in I$ - is the solution of the integral equation (14) with minimal norm in $L_2(I, R^m)$.

5. Existence of the solution

Solution of the differential equation (10) has a form

$$\xi(t)=\Phi(t,t_0)\xi_0+\int\limits_{t_0}^{t}\Phi(t,\tau)B_1(\tau)u(\tau)d\tau+\int\limits_{t_0}^{t}\Phi(t,\tau)\mu_1(\tau)d\tau,\ t\in I,$$

where $\Phi(t,\tau)=\overline{\theta}(t)\overline{\theta}^{-1}(\tau)$, $\overline{\theta}(t)$ - fundamental matrix of the solution of linear homogeneous system $\dot{y}=A_1(t)y$. Since $\xi(t_1)=\xi_1$, then

$$\xi_1=\xi(t_1)=\Phi(t_1,t_0)\xi_0+\int\limits_{t_0}^{t_1}\Phi(t_1,t)B_1(t)u(t)dt+\int\limits_{t_0}^{t_1}\Phi(t_1,t)\overline{\mu}(t)dt.$$

It follows that the desired control $u(t)\in U(t)$ is the solution of the following integral equation

$$\int\limits_{t_0}^{t_1}\Phi(t_0,t)B_1(t)u(t)dt=a=\Phi(t_0,t_1)\xi_1-\xi_0-\int\limits_{t_0}^{t_1}\Phi(t_0,t)\mu_1(t)dt. \tag{17}$$

Theorem 5.1. *Let the matrix*

$$W(t_0,t_1)=\int\limits_{t_0}^{t_1}\Phi(t_0,t)B_1(t)B_1^*(t)\Phi^*(t_0,t)dt$$

of $(n+m_2)\times(n+m_2)$ order be positively defined. Then the control $u(\cdot)\in L_2(I,R^m)$ transfers the trajectory of the system (10) from any initial point $\xi_0\in R^{n+m_2}$ to any final point $\xi_1\in R^{n+m_2}$ if and only if

$$\begin{aligned}u(t)\in\Lambda=\{u(\cdot)\in L_2(I,R^m)/\ u(t)=v(t)+T_1(t)\xi_0+T_2(t)\xi_1+\\+N_1(t)z(t_1,v)+\mu_2(t),\ \forall v,\ v(\cdot)\in L_2(I,R^m)\},\end{aligned} \tag{18}$$

where

$$T_1(t)=-B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1),\ T_2(t)=B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),$$
$$N_1(t)=-B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),\ \mu_2(t)=-B_1^*(t)\Phi^*(t_0,t)\cdot$$
$$W^{-1}(t_0,t_1)(\int\limits_{t_0}^{t_1}\Phi(t_0,t)\mu_1(t)dt),$$

function $z(t,v)$, $t\in I$ - is the solution of differential equation

$$\dot{z}=A_1(t)z+B_1(t)v,\ z(t_0)=0,\ v(\cdot)\in L_2(I,R^m). \tag{19}$$

The solution of differential equation (10), which corresponds to the control $u(t)\in\Lambda$, is defined by formula

$$\xi(t)=z(t,v)+E_1(t)\xi_0+E_2(t)\xi_1+\mu_3(t)+N_2(t)z(t_1,v),\ t\in I, \tag{20}$$

where

$$E_1(t)=\Phi(t,t_0)W(t,t_1)W^{-1}(t_0,t_1),\ E_2(t)=\Phi(t,t_0)W(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),$$
$$\mu_3=\int\limits_{t_0}^{t}\Phi(t,\tau)\mu_1(\tau)d\tau-\Phi(t,t_0)W(t_0,t)W^{-1}(t_0,t_1)\int\limits_{t_0}^{t_1}\Phi(t_0,t)\mu_1(t)dt,$$
$$N_2(t)=-\Phi(t_1,t_0)W(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),\ t\in I.$$

Proof. As follows from the Theorem 4.1, integral equation (17) has a solution if and only if the matrix (see (15))

$$C(t_0,t_1)=\int_{t_0}^{t_1} K(t_0,t)K^*(t_0,t)dt=\int_{t_0}^{t_1} \Phi(t_0,t)B_1(t)B_1^*(t)\Phi^*(t_0,t)dt=W(t_0,t_1)$$

of $(n+m_2)\times(n+m_2)$ order is positively defined, where $K(t_0,t)=\Phi(t_0,t)B_1(t)$, $t\in I$. Hence, set $\Lambda\neq\Phi$, where Φ - is empty set. As follows from the Theorem 4.2, general solution of integral equation (17) has a form (see (16))

$$\begin{gathered}u(t)=B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)a+v(t)-B_1^*(t)\Phi^*(t_0,t)W^{-1}(t_0,t_1)\cdot\\ \cdot\int_{t_0}^{t_1}\Phi(t_0,t)B(t)v(t)dt,\ \forall v,\ v(\cdot)\in L_2(I,R^m),\end{gathered}\tag{21}$$

where $a=\Phi(t_0,t_1)\xi_1-\xi_0-\int\limits_{t_0}^{t_1}\Phi(t_0,t)\mu_1(t)dt$.

Solution of the differential equation (19) can be represented as

$$z(t,v)=\int_{t_0}^{t_1}\Phi(t,t_0)z(t_0)+\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v(\tau)d\tau=\int_{t_0}^{t}\Phi(t,\tau)B_1(\tau)v(\tau)d\tau,$$

where $z(t_0)=0$. Hence,

$$z(t_1,v)=\int_{t_0}^{t_1}\Phi(t_1,t)B_1(t)v(t)=\Phi(t_1,t_0)\int_{t_0}^{t_1}\Phi(t_0,t)B_1(t)v(t)dt\tag{22}$$

From (21), (22) follows that required control $u(t)$, $t\in I$ is defined by the formula (18).

Let $u(t)\in\Lambda$. Then the solution of differential equation (10) can be represented as (20). □

As it follows from the Theorem 5.1 for the set of all possible controls, every element of which transfers trajectory of the system (10) from ξ_0 to ξ_1 is defined by formula (18). For solving the problem 2.1 (or the problem 3.1) we need to find control $u(t)\in U\times\Lambda$ from the intersection of sets U and Λ.

Hence, it is necessary to solve the following two problems: 1) intersection of U and Λ must be non-empty set i.e. $U\cap\Lambda\neq\Phi$; 2) we need to find points of the set $\Sigma=U\cap\Lambda$, when $\Sigma\neq\Phi$.

Solution of these problems can be reduced to solving the following optimization problem: minimize the functional

$$\begin{gathered}I(v,u,x_0,x_1,d,w)=\int\limits_{t_0}^{t_1}[|v(t)+T_1(t)\xi_0+T_2(t)\xi_1+N_1(t)z(t_1,v)+\\ +\mu_2(t)-u(t)|^2+|w(t)-L(t)P\xi(t)-l(t)|^2]dt\to\inf\end{gathered}\tag{23}$$

under the conditions

$$\dot z=A_1(t)z+B_1(t)v,\ z(t_0)=0,\ v(\cdot)\in L_2(I,R^m),\tag{24}$$

$$x_0\in S_0,\ x_1\in S_1,\ d\in D=\{d\in R^{m_1}/\ d\geq 0\},\tag{25}$$

$$u(t)\in U(t),\ w(t)\in W(t)=\{w(\cdot)\in L_2(I,R^s)/\ \omega(t)\leq w(t)\leq\varphi(t),\ \text{a.e. } t\in I\}.\tag{26}$$

Note that:

$$T_1(t)\xi_0=T_1(t)\begin{pmatrix}x_0\\ O_{m_21}\end{pmatrix}=(T_{11}(t),T_{12}(t))\begin{pmatrix}x_0\\ O_{m_21}\end{pmatrix}=T_{11}(t)x_0;$$

$$T_2(t)\xi_1 = T_2(t)\begin{pmatrix} x_1 \\ \overline{c} \end{pmatrix} = (T_{21}(t), T_{22}(t))\begin{pmatrix} x_1 \\ \overline{c} \end{pmatrix} = T_{21}(t)x_1 + T_{22}(t)\overline{c} =$$

$$= T_{21}(t)x_1 + (\Sigma_1(t), \Sigma_2(t))\begin{pmatrix} \overline{c}_1 - d \\ \overline{c}_2 \end{pmatrix} = T_{21}(t)x_1 - \Sigma_1(t)d + T_{22}\overline{c};$$

$$E_1(t)\xi_0 = (E_{11}(t), E_{12}(t))\begin{pmatrix} x_0 \\ O_{m_2 1} \end{pmatrix} = E_{11}(t)x_0,\ E_2(t)\xi_1 = (E_{21}(t),$$

$$E_{22}(t))\begin{pmatrix} x_1 \\ \overline{c} \end{pmatrix} = E_{21}(t)x_1 + (F_1(t), F_2(t))\begin{pmatrix} \overline{c}_1 - d \\ \overline{c}_2 \end{pmatrix} = E_{21}(t)x_1 - F_1(t)d +$$

$$E_{22}\overline{c},\ \overline{c} = (c_1, c_2, \ldots, c_{m_2}).$$

Now the optimization problem (23)-(26) can be written as:

$$\begin{gathered} I(v,u,x_0,x_1,d,w) = \int_{t_0}^{t_1} [|v(t) + T_{11}(t)x_0 + T_{21}(t)x_1 - \Sigma_1(t)d + \mu_4(t) + \\ +N_1(t)z(t_1,v) - u(t)|^2 + |w(t) - L(t)P[z(t,v) + E_{11}(t)x_0 + E_{21}(t)x_1 - \\ -F_1(t)d + \mu_5(t) + N_2(t)z(t_1,v)] - l(t)|^2]dt \to \inf \end{gathered} \tag{27}$$

under the conditions

$$\dot{z} = A_1(t)z + B_1(t)v(t),\ z(t_0) = 0,\ v(\cdot) \in L_2(I, R^m), \tag{28}$$

$$u(t) \in U(t),\ x_0 \in S_0,\ x_1 \in S_1,\ w(t) \in W(t),\ d \in D, \tag{29}$$

where $\mu_4(t) = \mu_2(t) + T_{22}(t)\overline{c}$, $\mu_5(t) = \mu_3(t) + E_{22}(t)\overline{c}$.

Let

$$\theta(t) = (v(t), u(t), x_0, x_1, d, w(t)) \in X = L_2(I, R^m) \times U \times S_0 \times S_1 \times D \times W \subset H =$$

$$= L_2(I, R^m) \times L_2(I, R^m) \times R^n \times R^n \times R^{m_1} \times L_2(I, R^s),\ q(t) = (\theta(t), z(t_1, v), z(t, v)),$$

$$F_0(q(t), t) = |v(t) + T_{11}(t)x_0 + T_{21}(t)x_1 - \Sigma_1(t)d + \mu_4(t) + N_1(t)z(t_1, v) -$$

$$-u(t)|^2 + |w(t) - L(t)P[z(t,v) + E_{11}(t)x_0 + E_{21}(t)x_1 - F_1(t)d + \mu_5(t) +$$

$$N_2(t)z(t_1, v)] - l(t)|^2.$$

Now the optimization problem (27)-(29) can be written as:

Minimize the functional

$$I(\theta) = \int_{t_0}^{t_1} F_0(q(t), t)dt \to \inf \tag{30}$$

under the conditions

$$\dot{z} = A_1(t)z + B_1(t)v,\ z(t_0) = 0,\ \theta(t) \in X \subset H,\ t \in I. \tag{31}$$

Theorem 5.2. *Let $S_0 \subset R^n$, $S_1 \subset R^n$, $U(t) \subset L_2(I, R^m)$ be bounded convex closed sets, and also:*

$$d \in D_{\rho_0} = \{d \in R^{m_1} /\ d \geq 0,\ |d| \leq \rho_0\},\ v(\cdot) \in L_2^{\rho}(I, R^m) = \{v(\cdot) \in L_2(I, R^m)/$$

$$\|v\| \leq \rho\},\ W(t_0, t_1) > 0,$$

where $\rho_0 > 0$, $\rho > 0$ - sufficiently large numbers.

Then:

1) the functional $I(\theta)$, $\theta \in X_1 = L_2^{\rho}(I, R^m) \times U \times S_0 \times S_1 \times D_{\rho_0} \times W$ is convex;

2) the functional $I(\theta)$, $\theta \in X_1$ reaches the lower bound on the set $X_1 \subset X \subset H$, $I(\theta_*) = I_* =$

$\inf_{\theta\in X_1} I(\theta)$, $\theta_* \in X_* \subset X_1$;
3) for the existence of program control it is necessary and sufficient that $I(\theta) = 0, \theta_* \in X_*$.

Proof. As follows from the hypothesis of Theorem, X_1 - is bounded closed convex set in H. Since the function $F_0(q,t) \geq 0$ is a quadratic form with respect to q, it can be represented in the form $F_0(q,t) = q^*Q(t)q + 2q^*a(t) + b(t)$, $t \in I$, where $Q(t) = Q^*(t) \geq 0$, $t \in I$. Then $\frac{\partial^2 F_0(q,t)}{\partial q^2} = 2Q(t) \geq 0$, $t \in I$. Hence, $F_0(q,t)$ is a convex function with respect to variable q. Solution $z(t, \alpha v_1 + (1-\alpha)v_2) = \alpha z(t,v_1) + (1-\alpha)z(t,v_2)$, $t \in I$. Since $F_0(\alpha q_1 + (1-\alpha)q_2, t) \leq \alpha F_0(q_1,t) + (1-\alpha)F_0(q_2,t)$, $t \in I$, $\forall q_1, q_2$, $\forall\alpha$, $\alpha \in [0,1]$, then

$$I(\alpha\theta_1 + (1-\alpha)\theta_2) = \int_{t_0}^{t_1} F_0(\alpha q_1(t) + (1-\alpha)q_2(t), t)dt \leq \alpha \int_{t_0}^{t_1} F_0(q_1(t),t)dt +$$

$$+(1-\alpha)\int_{t_0}^{t_1} F_0(q_2(t),t)dt = \alpha I(\theta_1) + (1-\alpha)I(\theta_2),\ \forall\theta_1,\theta_2 \in X_1,\ \forall\alpha,\ \alpha \in [0,1].$$

The first assertion is proved. The second statement follows from the weak lower semicontinuity of $I(\theta)$ on the weakly compact set X_1 in a reflexive space H.

The necessity of the third assertion of the theorem follows directly from Theorem 5.1 and $U \bigcap \Lambda \neq \Phi$. The sufficiency is follows from $I(\theta_*) = 0$. □

6. Program control

As it follows from Theorem 5.2 the program control can be found from the condition $I(\theta_*) = 0$, $\theta_* \in X_* \subset X_1 \subset X \subset H$. If $I(\theta_*) = 0$, then the desired program control

$$\begin{gathered} u_*(t) = v_*(t) + T_{11}(t)x_{o*} + T_{21}(t)x_{1*} - \Sigma_1(t)d_* + \mu_4(t) + \\ +N_1(t)z(t_1, v_*) \in U(t),\ t \in I, \end{gathered} \tag{32}$$

function

$$\begin{gathered} x_*(t) = P[z(t,v_*) + E_{11}(t)x_{0*} + E_{21}(t)x_{1*} - F_1(t)d_* + \mu_5(t) + \\ +N_2(t)z(t_1,v_*)] \in G(t),\ t \in I, \end{gathered} \tag{33}$$

where $\theta_*(t) = (v_*(t), u_*(t), x_{0*}, x_{1*}, d_*, w_*(t)) \in X_*$

Theorem 6.1. *Let $W(t_0,t_1) > 0$. Then the functional (23) under conditions (24)-(26) is continuously Frechet differentiable, and the functional gradient*

$$I'(\theta) = (I'_v(\theta), I'_u(\theta), I'_{x_0}(\theta), I'_{x_1}(\theta), I'_d(\theta), I'_w(\theta)) \in H$$

at any point $\theta \in X$ can be calculated by the formula

$$\begin{array}{ll} I'_v(\theta) = \dfrac{\partial F_0(q(t),t)}{\partial v} - B_1^*(t)\psi(t) & I'_u(\theta) = \dfrac{\partial F_0(q(t),t)}{\partial u} \\ I'_{x_0}(\theta) = \dfrac{\partial F_0(q(t),t)}{\partial x_0} & I'_{x_1}(\theta) = \dfrac{\partial F_0(q(t),t)}{\partial x_1} \\ I'_d(\theta) = \dfrac{\partial F_0(q(t),t)}{\partial d} & I'_w(\theta) = \dfrac{\partial F_0(q(t),t)}{\partial w} \end{array} \tag{34}$$

where $z(t,v)$ - solution of the differential equation (24), and the function $\psi(t)$, $t \in I$ - solution of the adjoint system

$$\dot{\psi} = \frac{\partial F_0(q(t),t)}{\partial z} - A_1^*(t)\psi(t),\ \psi(t_1) = -\int_{t_0}^{t_1} \frac{\partial F_0(q(t),t)}{\partial z(t_1)}dt. \tag{35}$$

Furthermore, the gradient of $I\ (\theta) \in H$ satisfies the Lipschitz condition

$$\|I'(\theta_1) - I(\theta_2)\| \leq l\|\theta_1 - \theta_2\|,\ \forall\theta_1, \theta_2 \in X. \tag{36}$$

Proof. Let $\theta(t) = (v(t), u(t), x_0, x_1, d, w(t)) \in X,\ \theta + \Delta\theta = (v(t) + h(t), u(t) + \Delta u(t), x_0 + \Delta x_0, x_1 + \Delta x_1, d + \Delta d, w(t) + \Delta w(t)) \in X$. Then the increment of the functional

$$\Delta I = I(\theta + \Delta\theta) - I(\theta) = \int_{t_0}^{t_1} [F_0(q(t) + \Delta q(t), t) - F_0(q(t), t)]dt,$$

where $q(t) + \Delta q(t) = (\theta(t) + \Delta\theta(t), z(t_1, v) + \Delta z(t_1, v), z(t, v) + \Delta z(t, v))$,

$$|\Delta z(t)| \leq \int_{t_0}^{t} \|\Phi(t,\tau)B_1(\tau)\||h(\tau)|d\tau \leq C_1 \int_{t_0}^{t_1} |h(t)|dt \leq C_2\|h\|_{L_2},\ t \in I,$$

where $C_1 = \sup\|\Phi(t,\tau)B_1(\tau)\|$, $t_0 \leq t, \tau \leq t_1$, $C_2 = C_1\sqrt{t_1 - t_0}$.

As $F_0(q,t)$ has continuous derivatives q and derivatives satisfy a Lipschitz conditions, then

$$\begin{aligned}\Delta I = \int_{t_0}^{t_1} \{h^*(t)[F_{0v}(q(t), t) - B_1^*(t)\psi(t)] + \Delta u^*(t)F_{0u}(q(t), t) + \Delta x_0^* F_{0x_0}(q(t), t) + \\ \Delta x_1^* F_{0x_1}(q(t), t) + \Delta d^* F_{0d}(q(t), t) + \Delta w^* F_{0w}(q(t), t) + \sum_{i=1}^{8} R_i,\end{aligned} \tag{37}$$

where $|R| = |\sum_{i=1}^{8} R_i| \leq \sum_{i=1}^{\infty} |R_i| \leq C_3\|\Delta\theta\|^2$. Then from (37) follows that the gradient $I'(\theta)$ is defined by the formula (34), where $\psi(t)$, $t \in I$ - solution of the equation (35).

Let $\theta_1 = \theta + \Delta\theta$, $\theta_2 = \theta$. Then from (34) follows

$$\begin{aligned}I'(\theta_1) - I'(\theta_2) = (F_{0v}(q(t) + \Delta q(t), t) - F_{0v}(q(t), t) - B_1^*(t)\Delta\psi(t), F_{0u}(q(t) + \\ + \Delta q(t), t) - F_{0u}(q(t), t), \int_{t_0}^{t_1} [F_{0x_0}(q(t) + \Delta q(t), t) - F_{0x_0}(q(t), t)]dt, \int_{t_0}^{t_1} [F_{0x_1}(q(t) + \\ + \Delta q(t), t) - F_{0x_1}(q(t), t)]dt, \int_{t_0}^{t_1} [F_{0d}(q(t) + \Delta q(t), t) - F_{0d}(q(t), t)]dt, \int_{t_0}^{t_1} [F_{0w}(q(t) + \\ + \Delta q(t), t) - F_{0w}(q(t), t)).\end{aligned}$$

Then

$$|I'(\theta_1) - I'(\theta_2)| \leq C_4|\Delta q(t)| + C_5|\Delta\psi(t)| + C_6\|\Delta q\|,$$

$$\|I'(\theta_1) - I'(\theta_2)\|^2 = \int_{t_0}^{t_1} |I'(\theta_1) - I'(\theta_2)|^2 dt \leq C_7\|\Delta q\|^2 + C_8 \int_{t_0}^{t_1} |\Delta\psi(t)|^2 dt, \tag{38}$$

where

$$|\Delta q(t)| \leq |h(t)| + |\Delta u(t)| + |\Delta x_0| + |\Delta x_1| + |\Delta d| + |\Delta w(t)| + |\Delta z(t_1)| + |\Delta z(t)|,$$

$$\|\Delta q\|^2 = \int_{t_0}^{t_1} |\Delta q(t)|^2 dt \leq C(\|h\|^2 + \|\Delta u\|^2 + \|\Delta x_0\|^2 + \|\Delta x_1\|^2 + \|\Delta d\|^2 + \|\Delta w\|^2).$$

As

$$\Delta\dot{\psi}(t) = F_{0z}(q(t) + \Delta q(t), t) - F_{0z}(q(t), t) - A_1^*(t)\Delta\psi(t),\ t \in I,$$

$$\Delta\psi(t_1) = -\int_{t_0}^{t_1} [F_{0z(t_1)}(q(t)+\Delta q(t),t) - F_{0z(t_1)}(q(t),t)]dt,$$

then by applying the Gronwall's lemma, we obtain

$$|\Delta\psi(t)| \le C_9\|\Delta q\|,\ t\in I. \tag{39}$$

From the estimates(38),(39) we get $\|I'(\theta_1) - I'(\theta_2)\| \le l_3\|\theta_1-\theta_2\|,\ \forall\theta_1,\theta_2\in X.$ □

To solve the optimization problem (30), (31) we construct sequences $\{\theta_n\}\subset X_1\subset X$ using the algorithm:

$$\begin{array}{ll} v_{n+1} = P_{L_2^\rho}[v_n - \alpha_n I'_v(\theta_n)], & u_{n+1} = P_U[u_n - \alpha_n I'_u(\theta_n)], \\ x_{0n+1} = P_{S_0}[x_{0n} - \alpha_n I'_{x_0}(\theta_n)], & x_{1n+1} = P_{S_1}[x_{1n} - \alpha_n I'_{x_1}(\theta_n)], \\ d_{n+1} = P_{D_{\rho_0}}[d_n - \alpha_n I'_d(\theta_n)], & w_{n+1} = P_W[w_n - \alpha_n I'_w(\theta_n)]. \end{array} \tag{40}$$

$$n=0,1,2,\ldots,\ 0<\epsilon_0\le\alpha_n\le\frac{2}{l+2\epsilon_1},\ \epsilon_1>0$$

where $l=const>0$ - Lipschitz constant from (36).

Theorem 6.2. *Assume the conditions of Theorem 5.2 and that the sequence $\{\theta_n\}\subset X_1$ determined by the relations from (40). Then:*
1) the sequence $\{\theta_n\}\subset X_1$ is minimizing, $\lim_{n\to\infty} I(\theta_n) = I_* =$
$= \inf_{\theta\in X_1} I(\theta)$*;*
2) the sequence $\{\theta_n\}\subset X_1$ weakly converges to the set $X_*\subset X_1$, where $v_n \xrightarrow{weakly} v_*$, $u_n \xrightarrow{weakly} u_*$, $x_{0n} \xrightarrow{weakly} x_{0*}$, $x_{1n} \xrightarrow{weakly} x_{1*}$, $d_n \xrightarrow{weakly} d_*$, $w_n \xrightarrow{weakly} w_*$ when $n\to\infty$, $\theta_* = (v_*,u_*,x_{0*},x_{1*},d_*,w_*)\in X_* = \{\theta_*\in X_1/\ I(\theta_*) = I_* = \inf_{\theta\in X_1} I(\theta)\}$;
3) the following estimation of the convergence's rate $I(\theta_n) - I_* \le \frac{c_0}{n}$, $c_0 = const > 0$, $n = 1,2,\ldots isvalid$;
4) function $u_*(t)\in U$ - is the required program control if and only if $I(\theta_*)=0$, where $u_*(t)$, $t\in I$ - weak limit point of the sequence $\{u_n\}\subset U$.

Proof. From (40) considering properties of the projection point on the set, we get

$$<\theta_{n+1}-\theta_n+\alpha_n I'(\theta_n), \theta-\theta_{n+1}>_H\ \ge 0,\ \forall\theta,\ \theta\in X_1. \tag{41}$$

Hence, in particular when $\theta=\theta_n\in X_1$, we get

$$< I'(\theta_n), \theta_n-\theta_{n+1}>_H \ge \frac{1}{\alpha_n}\|\theta_n-\theta_{n+1}\|^2. \tag{42}$$

Since the functional $I(\theta)\in C^{1,1}(X_1)$, then the following inequality is true

$$I(\theta_1) - I(\theta_2) \ge < I'(\theta_1), \theta_1-\theta_2> -\frac{l}{2}\|\theta_1-\theta_2\|,\ \forall\theta_1,\theta_2\in X_1.$$

Hence, for $\theta_1=\theta_n$, $\theta_2=\theta_{n+1}$ we get

$$I(\theta_n) - I(\theta_{n+1}) \ge < I'(\theta_n), \theta_n-\theta_{n+1}> -\frac{l}{2}\|\theta_n-\theta_{n+1}\|,\ \forall\theta_n,\theta_{n+1}\in X_1. \tag{43}$$

From (43) considering (42) we get

$$I(\theta_n) - I(\theta_{n+1}) \ge (\frac{1}{\alpha_n}-\frac{l}{2})\|\theta_n-\theta_{n+1}\|^2 \ge \epsilon_1\|\theta_n-\theta_{n+1}\|^2, \tag{44}$$

where $\frac{1}{\alpha_n}-\frac{l}{2}\ge\epsilon_1$. From (44) follows that the numerical sequence $\{I(\theta_n)\}$ strictly decreasing. Since the value of the functional $I(\theta)$ is lower bounded, then the numerical sequence $\{I(\theta_n)\}$ converges. Hence $\lim_{n\to\infty}[I(\theta_n)-I(\theta_{n+1})]=0$. Then $\|\theta_n-\theta_{n+1}\|\to 0$ when $n\to\infty$.

Let's show that the sequence $\{\theta_n\} \subset X_1$ is minimizing. As the functional $I(\theta) \in C^{1,1}(X_1)$ is convex, next inequality is performed

$$I(\theta_2) - I(\theta_1) \leq < I'(\theta_2), \theta_2 - \theta_1 >, \ \forall \theta_1, \theta_2 \in X_1.$$

From this inequality when $\theta_2 = \theta_n$, $\theta_1 = \theta_* \in X_* \subset X_1$, $X_* \neq \Phi$, $\theta_n \in X_1$ we get

$$I(\theta_n) - I(\theta_*) \leq < I'(\theta_n), \theta_n - \theta_* > = < I'(\theta_n), \theta_n - \theta_{n+1} > - < I'(\theta_n), \theta_* - \theta_{n_1} > .$$

From (41) when $\theta = \theta_*$, we get

$$< I'(\theta_n), \theta_* - \theta_{n+1} > \geq \frac{1}{\alpha_n} < \theta_n - \theta_{n+1}, \theta_* - \theta_{n+1} > .$$

Then

$$I(\theta_n) - I(\theta_*) \leq < I'(\theta_n) - \frac{1}{\alpha_n}(\theta_* - \theta_{n+1}), \theta_n - \theta_{n+1} > \leq l_1 \|\theta_n - \theta_{n+1}\| \tag{45}$$

where $l_1 = const > 0$. As $\|\theta_n - \theta_{n+1}\| \to 0$ when $n \to \infty$, then from (45) follows that $\lim_{n\to\infty} I(\theta_n) = I(\theta_*) = I_* = \inf_{\theta \in X_1} I(\theta)$. This means that the sequence $\{\theta_n\} \subset X_1$ is minimizing.

Let's show that the sequence $\{\theta_n\} \subset X_1$ weakly converges to a point $\theta_* \in X$. In fact, set X_1 is weakly compact, $\{\theta_n\} \subset X_1$. Hence sequence $\{\theta_n\} \subset X_1$ has at least one subsequence $\{\theta_{k_m}\} \subset X_1$ such as $\theta_{k_m} \xrightarrow{weakly} \theta_*$ when $m \to \infty$, and besides $\theta_* \in X_1$. As $\{I(\theta_n)\}$ converges to $I(\theta_*)$, then $\{I(\theta_{k_m})\}$ also converges to $I(\theta_*)$. Since the functional is weakly lower semicontinuous on X_1, then

$$I(\theta_*) \leq \underline{\lim}_{m\to\infty} I(\theta_{k_m}) \leq \lim_{m\to\infty} I(\theta_{k_m}) = I(\theta_*), \ \theta_{k_m} \xrightarrow{\text{weakly}} \theta_* \text{ when } m \to \infty.$$

From this we get $\underline{\lim}_{m\to\infty} I(\theta_{k_m}) = I(\theta_*) = \inf_{\theta\in X_1} I(\theta)$. Thus, in the weak limit point θ_* of the sequence $\{\theta_n\} \subset X_1$ reaches the lower bound of the functional $I(\theta)$ on the set X_1.

From the inequality (44), (45) it follows that

$$a_n = I(\theta_n) - (\theta_*) \leq l_1\|\theta_n - \theta_{n+1}\|, \ a_n - a_{n+1} \geq \epsilon_1 \|\theta_n - \theta_{n+1}\|^2.$$

Thus, numerical sequence $\{\alpha_n\}$ satisfies the conditions

$$a_n > 0, \ a_n - a_{n+1} \geq A a_n^2, \ n = 1, 2, \ldots, \ A = \frac{\epsilon_1}{l_1^2}. \tag{46}$$

For numerical sequence $\{a_n\}$ which satisfies the inequality (46) we have the estimate

$$a_n < \frac{1}{A_n}, \ n = 1, 2, \ldots, \ I(\theta_n) - I(\theta_*) \leq \frac{c_0}{n}, \ c_0 = \frac{l_1^2}{\epsilon_1}.$$

The last assertion follows from Theorem 5.2. Desired program control is determined by the formula (32), trajectory of the system (1) under conditions (2)-(6) defined by the formula (33). □

7. Position control

With using of the program control (32) we can find the position control $u_*(x_*, t)$, $t \in I$.

Theorem 7.1. *Assume the conditions of Theorems 6.1,6.2 and besides:*
1) $x_{1*} = R_1 x_{0*}$, $d_* = R_2 x_{0*}$, $v_*(t) = H(t) x_{0*}$, *where* $R_1, R_2, H(t)$ *- matrices of* $n \times n$ *order,* $m_1 \times n$, $m \times n$ *respectively;*
2) the value $I(\theta_*) = 0$;

3) the matrix $\Sigma(t) = P[\Phi(t,t_0)\Gamma(t) + E_{11}(t) + E_{21}(t)R_1 - F_1(t)R_2 +$ $+N_2(t)\Phi(t_0,t_1)\Gamma(t_1)]$ *of* $n \times n$ *order nonsingular, where* $\Gamma(t) = \int\limits_{t_0}^{t} \Phi(t_0,\tau)\cdot B_1(\tau)H(\tau)d\tau,\ t \in I$.

Then position control $u_*(x_*,t) = K(t)x_*(t) + \mu_6(t)$ *where*

$$K(t) = [H(t) + T_{11}(t) + T_{21}(t)R_1 - \Sigma_1(t)R_2 + N_1(t)\Phi(t_0,t_1)\Gamma(t_1)]\Sigma^{-1}(t)$$

$$\mu_6 = \mu_4 - K(t)P\mu_5(t),\ t \in I.$$

Proof. Control $u_*(t),\ t \in I$ from (32) can be represented in following form $u_*(t) = \overline{u}_*(t) + \mu_4(t)$, where $\overline{u}_* = v_*(t) + T_{11}x_{0*} + T_{21}(t)x_{1*} - \Sigma_1(t)d_* + N_1(t)z(t_1,v_*),\ t \in I$. Similarly, the function $x_*(t),\ t \in I$ from (33) can be represented as $x_*(t) = \overline{x}_*(t) + P\mu_5(t)$, where $\overline{x}_*(t) = P[z(t,v_*) + E_{11}(t)x_{0*} + E_{21}(t)x_{1*} - F_1(t)d_* + N_2(t)z(t_1,v_*)],\ t \in I$. Under the condition of Theorem functions $\overline{u}_*(t),\ \overline{x}_*(t),\ t \in I$ equal:

$$\overline{u}_*(t) = [H(t) + T_{11}(t) + T_{21}(t)R_1 - \Sigma_1(t)R_2 + N_1(t)\Phi(t_0,t_1)\Gamma(t_1)]x_{0*},\ t \in I,$$

$$\overline{x}_*(t) = \{P[\Phi(t,t_0)\Gamma(t) + E_{11}(t) + E_{21}(t)R_1 - F_1(t)R_2 + N_2(t)\Phi(t_0,t_1)\Gamma(t_1)]\}x_{0*} =$$

$$\Sigma(t)x_{0*},\ t \in I.$$

As $x_{0*} = \Sigma^{-1}(t)\overline{x}_*(t)$, then $\overline{u}_*(t) = K(t)\overline{x}_* = K(t)[x_*(t) - P\mu_5(t)] = K(t)x_*(t) - K(t)P\mu_5(t)$. Then $u_*(t) = K(t)x_*(t) - K(t)P\mu_5(t) + \mu_4(t) = K(t)x_*(t) + \mu_6(t),\ t \in I$. □

8. Optimal high-speed performance. Example

Let $t_{1*} > t_0$ be the minimal value of t_1 for which $I(\theta_*) = 0$ when $t_1 = t_{1*}$. It is necessary to find $u_*(t),\ t \in [t_0,t_{1*}],\ x_*(t) = x_*(t,u_*),\ t \in [t_0,t_{1*}]$ such as:

1) $u_*(t) \in U(t),\ t \in [t_0,t_{1*}]$; 2) $x_{0*} \in S_0,\ x_{1*} \in S_1$; 3) $x_*(t) \in G(t),\ t \in [t_0,t_{1*}]$; 4) $g_j(x_*,u_*) \le c_j,\ j = \overline{1,m_1}$; $g_j(x_*,u_*) = c_j,\ j = \overline{m_1+1,m_2}$.

To solve the optimal control problem it is necessary to solve the controllability problem for the values $t_{11}, t_{12}, \ldots$, where $t_1 > t_{11} > t_{12} > \ldots$.

Let the problem be solved for the given value $t_1 > t_0$. Let us choose $t_{11} = t_1/2$. Using the algorithm we will find $u_*(t),\ x_*(t),\ t \in [t_0,t_{11}]$. If for a given pair the value $I(\theta_*) = 0$, then we choose $t_{12} = t_1/4$ and so on. In case the given pair $I(\theta_*) > 0$, we choose $t_{12} = 3t_1/4$ and so on. Using given algorithm after finite amount of steps we will get approximate solution of the optimal high-speed performance problem with the needed accuracy.

Example. Minimize the functional

$$I(u,t_1) = \int\limits_0^{t_1} 1 \cdot dt = t_1 \to \inf \tag{47}$$

under conditions

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ t \in I = [0,t_1] \tag{48}$$

$$x_1(0) = x_{10},\ x_2(0) = x_{20},\ x_1(t_1) = x_{11},\ x_2(t_1) = x_{21} \tag{49}$$

$$u(t) \in U = \{u(\cdot) \in L_2(I,R^1) /\ -1 \le u(t) \le +1 \text{ a.e. } t \in I\}. \tag{50}$$

For the given example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} x_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}.$$

Program control. Consider the problem of controllability for control (48) with boundary conditions (49) when $u(\cdot) \in L_2(I, R_1)$. As

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \overline{\theta}(t),\ \overline{\theta}^{-1}(t) = e^{-At} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\ \Phi(t,\tau) = e^{A(t-\tau)},$$

then the matrices

$$W(0,t_1) = \int_0^{t_1} e^{-At}BB^*e^{-A^*t}dt = \begin{pmatrix} t_1^3/3 & -t_1^2/2 \\ -t_1^2/2 & t_1 \end{pmatrix},\ t_1 > 0,$$

$$W(0,t) = \begin{pmatrix} t^3/3 & -t^2/2 \\ -t^2/2 & t \end{pmatrix},\ W(t,t_1) = \begin{pmatrix} (t_1^3 - t^3)/3 & (-t_1^2 + t^2)/2 \\ (-t_1^2 + t^2)/2 & t_1 - t \end{pmatrix},$$

$$W^{-1}(0,t_1) = \begin{pmatrix} 12/t_1^3 & 6/t_1^2 \\ 6/t_1^2 & 4/t_1 \end{pmatrix},\ T_1(t) = -B^*e^{-A^*t}W^{-1}(0,t_1) = \left(\frac{12t}{t_1^3} - \frac{6}{t_1^2}, \frac{6t}{t_1^2} - \frac{4}{t_1}\right),$$

$$T_2(t) = B^*e^{-A^*t}W^{-1}(0,t_1)e^{-At_1} = \left(-\frac{12t}{t_1^3} + \frac{6}{t_1^2}, \frac{6t}{t_1^2} - \frac{2}{t_1}\right),$$

$$N_1(t) = -B^*e^{-A^*t}W^{-1}(0,t_1)e^{-At_1} = \left(\frac{12t}{t_1^3} - \frac{6}{t_1^2}, -\frac{6t}{t_1^2} + \frac{2}{t_1}\right),$$

As follows from Theorem 5.1, control

$$u(t) \in \Lambda = \{u(\cdot) \in L_2(I, R^1)/\ u(t) = v(t) + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1, v) =$$
$$v(t) + \left(\frac{12t - 6t_1}{t_1^2}\right)x_{10} + \left(\frac{6t - 4t_1}{t_1^2}\right)x_{20} + \left(\frac{-12t + 6t_1}{t_1^3}\right)x_{11} + \left(\frac{6t - 2t_1}{t_1^2}\right)x_{21} +$$
$$\left(\frac{12t - 6t_1}{t_1^3}\right)z_1(t_1, v) + \left(\frac{-6t + 2t_1}{t_1^2}\right)z_2(t_1, v).\ \forall v,\ v(\cdot) \in L_2(I, R^1)\}.$$

As

$$E_1(t) = e^{At}W(t,t_1)W^{-1}(0,t_1) = \begin{pmatrix} \dfrac{t_1^3 + 2t^3 - 3t^2t_1}{t_1^3} & \dfrac{t^3 + tt_1^2 - 2t^2t_1}{t_1^2} \\ \dfrac{6t^2 - 6tt_1}{t_1^3} & \dfrac{t_1^2 + 3t^2 - 4tt_1}{t_1^2} \end{pmatrix},$$

$$E_2(t) = e^{At}W(0,t)W^{-1}(0,t_1)e^{-At_1} = \begin{pmatrix} \dfrac{-2t^3 + 3t_1t^2}{t_1^3} & \dfrac{t^3 - t_1t^2}{t_1^2} \\ \dfrac{-6t^2 + 6tt_1}{t_1^3} & \dfrac{3t^2 - 2tt_1}{t_1^2} \end{pmatrix},$$

$$N_2(t) = -e^{At}W(0,t)W^{-1}(0,t_1)e^{-At_1} = \begin{pmatrix} \dfrac{2t^3 - 3t_1t^2}{t_1^3} & \dfrac{-t^3 + t_1t^2}{t_1^2} \\ \dfrac{6t^2 - 6tt_1}{t_1^3} & \dfrac{-3t^2 + 2tt_1}{t_1^2} \end{pmatrix},$$

then

$$x_1(t) = z_1(t,v) + \left(\frac{t_1^3 + 2t^3 - 3t^2t_1}{t_1^3}\right)x_{10} + \left(\frac{t^3 + tt_1^2 - 2t^2t_1}{t_1^2}\right)x_{20} +$$
$$\left(\frac{-2t^3 + 3t_1t^2}{t_1^3}\right)x_{11} + \left(\frac{t^3 - t_1t^2}{t_1^2}\right)x_{21} + \left(\frac{2t^3 - 3t^2t_1}{t_1^3}\right)z_1(t_1,v) + \left(\frac{-t^3 + t_1t^2}{t_1^2}\right)z_2(t_1,v),$$
$$x_2(t) = z_2(t) + \left(\frac{6t^2 - 6tt_1}{t_1^3}\right)x_{10} + \left(\frac{t_1^2 + 3t^2 - 4tt_1}{t_1^2}\right)x_{20} + \left(\frac{-6t^2 + 6tt_1}{t_1^3}\right)x_{11} +$$
$$\left(\frac{3t^2 - 2tt_1}{t_1^2}\right)x_{21} + \left(\frac{6t^2 - 6tt_1}{t_1^3}\right)z_1(t_1,v) + \left(\frac{-3t^2 + 2tt_1}{t_1^2}\right)z_2(t_1,v),\ t \in I.$$

where $\dot{z}_1 = z_2$, $\dot{z}_2 = v$, $z_1(0) = 0$, $z_2(0) = 0$, $v\cdot \in L_2(I, R^1)$, $t \in I = [0, t_1]$.

To determine the program control of (48)-(50) it is necessary to find the control from intersection of the sets $\Lambda \bigcap U$. Optimization problem (23)-(26) with fixed $x_0 \in R^2$, $x_1 \in R^2$ and no phase and integral restrictions can be written as

$$I(v,u) = \int_0^{t_1} |v(t) + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1,v) - u(t)|^2 dt \to \inf \tag{51}$$

under conditions

$$\dot{z}_1 = z_2,\ \dot{z}_2 = v,\ z_1(0) = 0,\ z_2(0) = 0,\ v\cdot \in L_2(I, R^1),\ u(t) \in U. \tag{52}$$

Function $F_0(q(t),t) = |v(t) + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1,v) - u(t)|^2$, when $q(t) = (v(t), u(t), z(t_1,v))$. As follows from Theorem 6.1, gradient $I'(v,u) = (I'_v(v,u), I'_u(v,u))$, where

$$I'_v(v,u) = \frac{\partial F_0}{\partial v} - B^*\psi(t) = 2[v(t) + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1,v) - u(t)] - \psi_2(t),$$

$$I'_u(v,u) = \frac{\partial F_0}{\partial u} = -2[v(t) + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1,v)].$$

As $\partial F_0/\partial z = 0$, then $\dot{\psi}_1 = 0$, $\dot{\psi}_2 = -\psi_1$, $\psi(t_1) = -\int_0^{t_1} \frac{\partial F_0}{\partial z(t_1)} dt =$
$-\int_0^{t_1} 2N_1^*(t)[v(t) + T_1(t)x_0 + T_2(t)x_1 + N_1(t)z(t_1,v) - u(t)]dt.$

Minimizing sequences $\{v_n\}$, $\{u_n\}$ are equal to:

$$v_{n+1} = v_n - \alpha_n I'_v(v_n,u_n),\ u_{n+1} = P_U[u_n - \alpha_n I'_u(v_n,u_n)],\ n = 0,1,2,\ldots. \tag{53}$$

The solution of the optimal high-speed performance problem when $x_{10} = 1$, $x_{20} = x_{11} = x_{21} = 0$.

A. Let us choose the value $t_1 = 8$. After solving the optimization problem (51), (52) by constructing a minimizing sequence (53) we will find

$$u_*(t) = v_*(t) = \begin{cases} -1, & \text{if } 0 \le t < \frac{17}{8}; \\ +1, & \text{if } \frac{17}{8} \le t < \frac{49}{8}; \\ -1, & \text{if } \frac{49}{8} \le t \le 8. \end{cases}$$

value $I(v_*, u_*) = 0$.

B. Let us choose $t_1 = \frac{8}{2} = 4$. For the value $t_1 = 4$ optimal solution of the problem (51),(52) will have a form

$$u_{**}(t) = v_{**}(t) = \begin{cases} -1, & \text{if } 0 \le t < \frac{5}{4}; \\ +1, & \text{if } \frac{5}{4} \le t < \frac{13}{4}; \\ -1, & \text{if } \frac{13}{4} \le t \le 4. \end{cases}$$

value $I(v_{**}, u_{**}) = 0$.

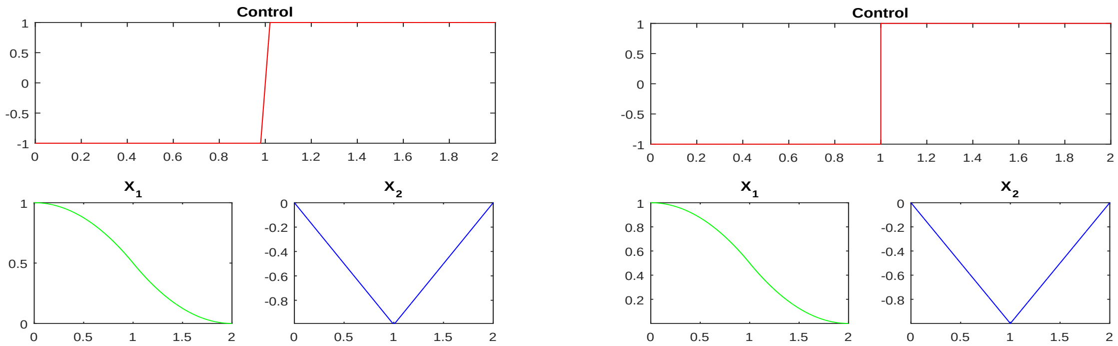
C. Let us choose $t_1 = \frac{4}{2} = 2$. For the value $t_1 = 2$ optimal solution of the problem (51),(52) will have a form

$$u_{***}(t) = v_{**}(t) = \begin{cases} -1, & \text{if } 0 \le t < 1; \\ +1, & \text{if } 1 \le t \le 2. \end{cases}$$

value $I(v_{***}, u_{***}) = 0$. Optimal trajectory for the problem (47)-(50)

$$x_{1*} = \begin{cases} 1 - \frac{t^2}{2}, & 0 \le t < 1; \\ \frac{t^2}{2} - 2t + 2, & 1 \le t \le 2, \end{cases} \quad x_{2*} = \begin{cases} -t, & 0 \le t < 1; \\ t - 2, & 1 \le t \le 2, \end{cases}$$

The optimal solution of this problem was implemented in the Matlab. The following graphs show that result is very similar to the solution which was found using Maximum principle of Pontryagin.



a) approximate solution
(Principle of immersion)

b) analytical solution
(Pontryagin's maximum principle)

Figure 1. Solution of the problem

9. Conclusion

One of the complex and unsolved problems of control theory is the existence of solutions of the boundary value problem of optimal control under phase and integral constraints. To solve the problems of existence of solutions it is necessary to create a general theory of controllability of dynamical systems. This work is devoted to solving of the control problem for complex dynamical systems with boundary conditions and restrictions.

The main results obtained in the work are: allocation of the set of program and position controls for the process described by linear ordinary differential equations in the absence of restrictions on the control values by building the solution of a Fredholm integral equation of the first kind; definition of program and positional control, as well as solving the problems of optimal high-speed performance in the presence of constraints on the values of control and phase and integral constraints; reduction of the original boundary value problem with constraints to a special initial optimal control problem and constructing of minimizing sequences; solving of optimal control problem by successively narrowing the set of admissible controls.

Scientific novelty of the results is that it was created a general theory of controllability and optimal performance for a linear ordinary differential equation.

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